ON THE INTEGRATION OF THE DIFFUSION EQUATION WITH BOUNDARY CONDITIONS(1)

BY

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1. **Introduction.** The purpose of this paper is to integrate the diffusion equation

$$(1.1)\qquad \begin{aligned} u_t &= Lu, \qquad t > 0,\\ u(0, x) &= f(x) \end{aligned}$$

where

$$(1.2)\qquad Lu = e(x)^{-1}[(a^{ij}(x)u_j)_i + (b^i(x)u)_i + c(x)u]$$

and x ranges over a domain Δ in the real m-dimensional Euclidean space R_m; here the summation convention is used, the subscript i denoting differentiation with respect to the ith component of x. All coefficients are assumed to be real-valued. On imposing boundary conditions of the type

$$(1.3)\qquad \sigma(x)[a^{nj}(x)u_j + b^n(x)u] + \tau(x)u = 0,$$

the initial-value problem is shown to have a unique positivity preserving solution; here $a^{nj} = a^{ij}n^i$, $b^n = b^i n^i$ (the n^i being components of the outer normal to Δ), σ, $\tau \geqq 0$, and $\sigma^2 + \tau^2 = 1$. Solutions in the form of strongly continuous semi-groups of positive operators (see [4]) are obtained in the L_p spaces over Δ with weight factor e, $1 \leqq p \leqq 2$.

The one-spatial variable L_1 problem has been treated in a definitive fashion by W. Feller [2]; for an L_2 treatment of this case in the spirit of the present paper the reader is referred to [11]. The study of the several-spatial variable problem from the point of view of semi-groups of operators was initiated by K. Yosida. He established the existence of positivity preserving solutions in L_1 for the diffusion equation on a compact Riemannian space without boundaries [12; 13]. A semigroup solution for the boundary value problem (1.3) was first obtained by S. Ito [5] by means of the fundamental solution for (1.1) and (1.3). This method has the advantage of not requiring growth conditions of the coefficients at infinity. However, concomitant with this is the disadvantage of there being several possible extensions of the differential operator (1.2), defined on smooth functions satisfying the boundary conditions (1.3), which generate semi-group solutions. Ito does not give a direct characterization of the domain of the infinitesimal generator for the semi-group whose existence he establishes.

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Our approach is entirely different from that of Ito, being based on a combination of the theory of dissipative operators as developed by the author [10; 11] and the theory of symmetric positive differential operators with local boundary conditions due to K. O. Friedrichs [3]. Actually a simplified version of Friedrichs' work, available in a paper by P. D. Lax and R. S. Phillips [6], suffices for our purposes. We are able to treat the diffusion equation over an unbounded domain Δ whose boundary $\dot{\Delta}$ contains a compact set of edges, while imposing only mild conditions on the smoothness and growth of the coefficients. As regards the boundary conditions (1.3), we assume merely that σ and τ are of class C^1 on $\dot{\Delta}$ except perhaps on the above mentioned edges where they need not even be continuous. Our conditions on the smoothness of the coefficients and the boundary are roughly one order of differentiability less than the corresponding conditions of Ito and in addition we permit edges where Ito does not. Under these hypotheses we are able to characterize the domain of the infinitesimal generator of a strongly continuous semi-group solution of positive contraction operators $[S(t)]$ in the L_p $(1 \leqq p \leqq 2)$ settings. In the L_2 case we show that $S(t)$ has an analytic extension in t into the sector $-\pi/4 < \arg t < \pi/4$. Throughout our development we have avoided the delicate problem of the smoothness of the solution near the boundary. Instead there is a heavy reliance on the existence of a strong solution for the equation $\lambda u - Lu = f$, $\lambda > 0$.

A few words about the nature of a semi-group solution to (1.1) are perhaps in order at this point. A one-parameter family of bounded linear operators $[S(t); t \geqq 0]$ on a Banach space $\mathfrak{Y}$ is called a strongly continuous semi-group of operators if

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$, t_1, $t_2 \geqq 0$,

(ii) $S(t)y$ is continuous in $t \geqq 0$ for each y in $\mathfrak{Y}$.

The *infinitesimal generator* A of the semi-group $[S(t)]$ is defined by

$$\lim_{t \to 0+} t^{-1}[S(t)y - y] = Ay \tag{1.4}$$

with its domain $\mathfrak{D}(A)$ consisting of all y in $\mathfrak{Y}$ for which this limit exists. It can be shown that A is a closed linear operator with dense domain and that for y in $\mathfrak{D}(A)$

$$dS(t)y/dt = AS(t)y = S(t)Ay, \qquad t \geqq 0. \tag{1.4}$$

This, then, is the sense in which the function $S(t)y$ satisfies a differential equation of the form (1.1) with L replaced by A. Given a closed linear operator A, suppose $(\lambda I - A)$ is one-to-one with range equal to all of $\mathfrak{Y}$. Then λ is said to belong to the resolvent set of A and the inverse of $(\lambda I - A)$ is called the resolvent operator of A at λ and is denoted by $R(\lambda; A)$.

The Hille-Yosida theorem [4, Theorem 12.3.1] states that a closed linear operator A with dense domain generates a strongly continuous semi-group of contraction operators if and only if

$$(1.6) \qquad |R(\lambda; A)| \leqq 1/\lambda, \qquad \lambda > 0.$$

The corresponding semi-group operators will be positive if and only if the resolvent operators $R(\lambda; A)$, $\lambda > 0$, are positive [4, Theorem 11.7.2]. In the special case where $\mathfrak{Y}$ is a Hilbert space with inner product (y^1, y^2) a somewhat simpler criterion applies. A linear operator on a Hilbert space is called *dissipative* if

$$(1.7) \qquad (Ay, y) + (y, Ay) \leqq 0, \qquad y \in \mathfrak{D}(A),$$

and *maximal dissipative* if it is not the proper restriction of any other dissipative operator. If A is maximal dissipative with dense domain (A is necessarily closed), then $\lambda > 0$ belongs to its resolvent set and $R(\lambda; A)$ is of norm $\leqq 1/\lambda$ (see [10]). As a consequence, a maximal dissipative operator with dense domain generates a strongly continuous semi-group of contraction operators on a Hilbert space.

In the first part of this paper dealing with the L_2-theory we proceed to define the operator L of (1.1) so that it is maximal dissipative and then by a study of the resolvent $R(\lambda; L)$ we show that the associated semi-group of operators is positive and has an analytic extension. In the second part of the paper which is concerned with the L_1-theory, the L_2-results are adapted so as to verify (1.6) and the positivity of the resolvent for the smallest closed L_1-extension of the largest L_1-restriction of the above defined L_2-operator L. The corresponding result for L_p $(1 < p < 2)$ is then an immediate corollary of the preceding L_1 and L_2 developments.

2. **Hypotheses and results.** In order to simplify the statements of our results we now list the essential hypotheses which will be needed in the following development. In general it will be found that the unprimed conditions are required in the L_2-theory whereas their primed counterparts are required in the L_1-theory.

H_1 (Domain conditions). *The domain $\Delta \subset R_m$ need not be bounded. However, except for points belonging to a compact subset F of the boundary $\dot{\Delta}$, to each $x \in \dot{\Delta}$ there is a neighborhood patch N_x which maps into a hemisphere*

$$\sum (x^i)^2 < 1 \quad \textit{and} \quad x^m > 0,$$

with the boundary portion of the patch mapping into $x^m = 0$. This map and its inverse is assumed to be one-to-one and of class C^2 on $N_x \cap \bar{\Delta}$. To each $x \in F$, there is a neighborhood patch N_x which maps into a polyhedron, the boundary portion of N_x going into the faces of the polyhedron. This map and its inverse is assumed to be one-to-one and of class C^2 in $N_x \cap \bar{\Delta}$ except perhaps along the edges of the polyhedron where they need only be of class C^1.

H_2 (Coefficient smoothness conditions). *The coefficients e, a^{ij}, and b^i are continuous and piecewise of class C^1 on $\bar{\Delta}$ and c is piecewise continuous on $\bar{\Delta}$.*

H_2'. *The coefficients e, a^{ij}, and b^i are of class C^1 on $\bar{\Delta}$ and c is of class C^0 on*

$\bar{\Delta}$. *In addition c and the first partials of e, a^{ij}, and b^i are Hölder continuous on compact subsets of* Δ.

H_3 (Dissipative condition). *For each $x \in \Delta$ the coefficient $e > 0$, the matrix $(a^{ij}) > 0$, the matrix*

$$\begin{pmatrix} c & b^j \\ b^i & -a^{ij} \end{pmatrix} \leqq 0 \tag{2.1}$$

and the matrix

$$\begin{pmatrix} -c & 0 \\ 0 & a^{ij} \end{pmatrix} \leqq K \begin{pmatrix} e & 0 \\ 0 & a^{ij} \end{pmatrix} \tag{2.2}$$

for some constant K. The matrices represented in (2.1) *and* (2.2) *are in blocks, the upper left being* 1×1 *and the lower right being* $m \times m$.

Remark. Condition (2.1) can sometimes be satisfied if $c(x)$ is replaced by $c'(x) = c(x) - \omega$. This has the effect of changing $S(t)$ from a contraction operator to an operator of bound less than or equal to $\exp(\omega t)$. Setting

$$\begin{aligned} \bar{\alpha}(x) &= \text{maximum eigenvalue of } (a^{ij}(x)), \\ \alpha(x) &= \text{minimum eigenvalue of } (a^{ij}(x)), \\ |b|(x) &= [\textstyle\sum (b^i(x))^2]^{1/2}, \end{aligned} \tag{2.3}$$

it is readily seen that (2.1) and (2.2) will be satisfied if

$$-c\alpha \geqq |b|^2 \quad \text{and} \quad |c| \leqq Ke, \qquad x \in \Delta;$$

on the other hand (2.1) implies

$$-c\bar{\alpha} \geqq |b|^2. \tag{2.4}$$

H_4 (Growth conditions on the coefficients). *Setting $r^2 = \sum (x^i)^2$, the coefficients satisfy the conditions*

$$\begin{aligned} |b|/e &= O(r), \\ \bar{\alpha}/e &= O(r^2), \\ |b|/(\alpha e^{1/2}) &= O(1), \end{aligned} \qquad \text{as } r \to \infty\ (^2).$$

H_4'.

$$\begin{aligned} |b|/e &= O(r/\log r), \\ \bar{\alpha}/e &= O([r/\log r]^2), \\ |b|/(\alpha e^{1/2}) &= O(1), \\ e \text{ and } |a^{ij}_j|/e &= O(r^k) \quad \text{for some } k, \end{aligned} \qquad \text{as } r \to \infty\ (^2).$$

(2) In the presence of H_3 it is clear that the condition on $|b|/e$ is implied by the condition on $\bar{\alpha}/e$.

H_5 (BOUNDARY CONDITIONS). *The factors σ and τ are of class C^1 on Δ except perhaps along F.*

We denote the space of measurable complex-valued functions over Δ with inner product

$$(y, z) = \int_\Delta e(x)y(x)(z(x))^- dx$$

by $L_2(\Delta, e)$, and the space of measurable complex-valued functions over Δ with norm

$$| y | = \int_\Delta e(x) | y(x) | dx$$

by $L_1(\Delta, e)$. Each of these spaces has a natural positive cone, namely the real non-negative valued functions in the respective spaces.

We now give two equivalent definitions for the operator L of (1.2) with boundary conditions (1.3) as an operator on $L_2(\Delta, e)$.

DEFINITION 2.1 (WEAK DEFINITION OF L). *A function u with strong first partials satisfying*

$$\int_\Delta [e | u |^2 + a^{ij}u_i u_j^-] dx < \infty \tag{2.5}$$

is contained in $\mathfrak{D}(L)$ if and only if there exists an $f \in L_2(\Delta, e)$ such that

$$\int_\Delta ef\bar{\phi}dx = \int_\Delta \{u[(a^{ij}\psi^j)_i - 2(b^i\phi)_i + (c + b^i_i)\phi]^- + u_i[a^{ij}\psi^j - a^{ij}\phi_j - b^i\phi]^-\} dx \tag{2.6}$$

for all smooth vector-valued functions (ϕ, ψ^i) with bounded support and which satisfy the boundary condition

$$\sigma(a^{ni}\psi^i - b^n\phi) + \tau\phi = 0, \qquad x \in \dot{\Delta}. \tag{2.7}$$

In this case $Lu = f$.

DEFINITION 2.2 (STRONG DEFINITION). *A function u with strong first partials satisfying* (2.5) *is contained in $\mathfrak{D}(L)$ if and only if there exists a sequence of smooth vector-valued functions $\{(u^k, v^{ik})\}$ with bounded support such that*

(i) $\sigma(a^{ni}v^{ik} + b^n u^k) + \tau u^k = 0, x \in \dot{\Delta}$,

(ii) $u^k \to u$ *in* $L_2(\Delta, e)$, $(\int_\Delta a^{ij}(u_i^k - u_i)(u_j^k - u_j)^- dx \to 0)$,

$$\int_\Delta a^{ij}(v^{ik} - u_i)(v^{jk} - u_j)^- dx \to 0,$$

$$e^{-1}[(a^{ij}v^{jk})_i + 2(b^i u^k)_i - b^i v^{ik} + (c - b^i_i)u^k] \to Lu \quad in \quad L_2(\Delta, e).$$

THEOREM 2.1. *Under the hypotheses* H_1, H_2, H_3, H_4, *and* H_5, *Definitions* 2.1 *and* 2.2 *define the same operator* L *on* $L_2(\Delta, e)$.

THEOREM 2.2. *Under the hypotheses* H_1, H_2, H_3, H_4, *and* H_5, *the operator* L *defined as in Definitions* 2.1 *and* 2.2 *is maximal dissipative with dense domain in* $L_2(\Delta, e)$ *and generates a strongly continuous semi-group of positive contraction operators* $[S(t)]$ *which has an analytic semi-group continuation throughout the sector* $-\pi/4 < \arg t < \pi/4$.

The corresponding operator in $L_1(\Delta, e)$ which we denote by K is defined as follows.

DEFINITION 2.3. *Let* L *denote the operator in* $L_2(\Delta, e)$ *defined by Definitions* 2.1 *and* 2.2, *and set*

$$K_0 u = Lu,$$

$$\mathfrak{D}(K_0) = [u;\, u \text{ and } Lu \text{ in } L_1(\Delta, e) \cap L_2(\Delta, e)].$$

The operator K *is the smallest closed extension of* K_0 *considered as an operator in* $L_1(\Delta, e)$.

THEOREM 2.3. *Let* $\bar{c} = \sup[c(x)/e(x);\, x \in \Delta]$. *Under the hypotheses* H_1, H_2', H_3, H_4', *and* H_5, *the operator* K *of Definition* 2.3 *is the infinitesimal generator of a strongly continuous semi-group of positive operators* $[S(t)]$ *in* $L_1(\Delta, e)$ *with* $|S(t)| \leqq \exp(\bar{c}t)$.

COROLLARY. *Let* K_p *denote the smallest closed extension of* K_0 *considered as an operator in* $L_p(\Delta, e)$. *Then for* $1 < p < 2$, K_p *is the infinitesimal generator of a strongly continuous semi-group of positive operators* $[S(t)]$ *in* $L_p(\Delta, e)$ *with* $|S(t)| \leqq \exp[(2/p - 1)\bar{c}t]$.

3. **The L_2-existence theory.** In this section we prove that the operator L is maximal dissipative with dense domain by employing the results of [6; 11]. The assertion of Theorem 2.1 turns out to be a by-product of this development.

We proceed as in [11] and study a first order differential operator M whose retract is L. The domain of M consists of all complex vector-valued functions $y = (u, v^1, \cdots, v^m)$ which are piecewise smooth on $\bar{\Delta}$, have bounded support, and satisfy the boundary condition

$$\sigma(a^{ni}v^i + bu) + \tau u = 0, \qquad x \in \dot{\Delta}. \tag{3.1}$$

The operator M is then defined as

$$My = E^{-1}[(A^i y)_i + By], \tag{3.2}$$

where

$$E = \begin{pmatrix} e & 0 \\ 0 & a^{ij} \end{pmatrix}, \qquad A^k = \begin{pmatrix} 2b^k & a^{kj} \\ a^{ik} & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} c - b^k_k & -b^j \\ -a^{ik}_k & -a^{ij} \end{pmatrix}.$$

We further define

$$D = B + B^* + A^k_k = \begin{pmatrix} 2c & -b^j \\ -b^i & -2a^{ij} \end{pmatrix}. \tag{3.4}$$

Making use of (2.1) it is readily verified that

$$P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \leqq -E^{-1}D; \tag{3.5}$$

this corresponds to condition (4.2) of [11].

We also introduce the Hilbert space $L_2(\Delta, E)$ of complex vector-valued functions with inner product defined as

$$(y, z) = \int_\Delta \langle Ey, z\rangle dx, \tag{3.6}$$

where for $y = (\eta^i)$ and $z = (\zeta^i)$ we have $\langle y, z\rangle = \eta^i\bar{\zeta}^i$. Obviously M can be considered an operator in $L_2(\Delta, E)$ and it is clear that $\mathfrak{D}(M)$ is dense in this space.

LEMMA 3.1. *Under the hypotheses* H_1, H_2, H_3, H_4, *and* H_5, *the smallest closed extension of* M, *say* $\overline{M}$, *is maximal dissipative with dense domain.*

Proof. The assertion of the lemma is essentially that of [6, Theorem 3.2]. However two things need be checked: (i) the boundary conditions (3.1) must be shown to be sufficiently smooth; and (ii) the set F together with the point at infinity must be shown to be unessential in the sense of [6, §5].

We first consider the boundary conditions. Setting

$$A^n = A^k n^k$$

we see that

$$A^n = \begin{pmatrix} 2b^n & a^{nj} \\ a^{in} & 0 \end{pmatrix},$$

which is obviously of rank two. A simple calculation shows that the boundary condition fills out a maximal negative subspace relative to the quadratic form $\langle A^n y, y\rangle$ at each point of $\dot{\Delta}$. Moreover A^n can be defined in the interior of the transformed boundary patch to be of constant rank by setting $A^n = A^m$. The only other requirement in the proof of [6, Theorem 3.2] is that in each boundary patch disjoint from F there exist a rotation $y = Uz$ of class C^1 such that the null space of A^n fills out the last $m-1$ components of z and the boundary conditions become $\zeta^0 = 0$. Now in the original y-coordinate system, the null space of A^n is spanned by vectors c which have a zero first component and which are orthogonal to $(0, a^{n1}, \cdots, a^{nm})$. Since the a^{ni} are of class C^1 it is clear that we can construct (at least locally) $m-1$ orthogonal vectors $c^2(x), \cdots, c^m(x)$ of class C^1 which span this null space. Writing

$$v^1 = (1, 0, \cdots, 0),$$
$$v^2 = \alpha^{-1}(0, a^{n1}, \cdots, a^{nm}) \quad \text{with} \quad \alpha^2 = \sum (a^{ni})^2,$$

we now define $c^0(x)$ and $c^1(x)$ by

$$[\alpha^2\sigma^2 + (\sigma b^n + \tau)^2]^{1/2} c^0 = (\sigma b^n + \tau)v^1 + \alpha\sigma v^2,$$
$$[\alpha^2\sigma^2 + (\sigma b^n + \tau)^2]^{1/2} c^1 = -\alpha\sigma v^1 + (\sigma b^n + \tau)v^2.$$

Since $\sigma^2 + \tau^2 = 1$ it is clear that $[\alpha^2\sigma^2 + (\sigma b^n + \tau)^2]$ can never vanish and hence that $c^0(x)$ and $c^1(x)$ are also of class C^1. Finally we note that the matrix with jth column equal to c^j satisfies the required conditions for the rotation U.

In order to show that M is essentially maximal dissipative it suffices to show that the adjoint operator M^* is dissipative. Thus given $z \in \mathfrak{D}(M^*)$, we construct a sequence of real-valued smooth scalar functions $\{\phi^\nu\}$ as follows: $\phi^\nu = 1$ at points of $\bar{\Delta}$ which are at a distance $> 1/\nu$ from F and $< \nu$ from the origin; $\phi^\nu = 0$ at points which are at a distance $< 1/(\nu+1)$ from F and $> \nu + 1$ from the origin; ϕ^ν is smoothly extended elsewhere so that near F its gradient is less than $2\nu^2$ and elsewhere its gradient is less than 2. It is clear that $\phi^\nu z \in \mathfrak{D}(M^*)$ and since this function is of bounded support and vanishes near F, the argument used in [6, Theorem 3.2] shows that

$$(M^*\phi^\nu z, \phi^\nu z) + (\phi^\nu z, M^*\phi^\nu z) - (D\phi^\nu z, \phi^\nu z) \leqq 0$$

for ν sufficiently large. Consequently M^* will be dissipative if it can be shown that

$$(3.7) \quad \begin{aligned} &(M^*z, z) + (z, M^*z) - (Dz, z) \\ &\qquad = \lim_{\nu\to\infty} [(M^*\phi^\nu z, \phi^\nu z) + (\phi^\nu z, M^*\phi^\nu z) - (D\phi^\nu z, \phi^\nu z)]; \end{aligned}$$

this, incidentally, is what is meant by $F \cup \{\infty\}$ being unessential. Now

$$(3.8) \quad \begin{aligned} &(M^*\phi^\nu z, \phi^\nu z) + (\phi^\nu z, M^*\phi^\nu z) - (D\phi^\nu z, \phi^\nu z) \\ &\qquad = [((\phi^\nu)^2 M^*z, z) + ((\phi^\nu)^2 z, M^*z) - ((\phi^\nu)^2 Dz, z)] - 2(\phi^\nu \phi_j^\nu A^j z, z). \end{aligned}$$

The bracketed expression in the right member of (3.8) obviously converges to the left member of (3.7). Hence (3.7) will be verified if it can be shown that

$$\liminf_{\nu\to\infty} \left| (\phi^\nu \phi_j^\nu A^j z, z) \right| = 0. \tag{3.9}$$

Suppose on the contrary that

$$\left| (\phi^\nu \phi_j^\nu A^j z, z) \right| \geqq \epsilon > 0 \tag{3.10}$$

for ν sufficiently large. Set

$$\begin{aligned} \Delta_\nu^1 &= [x; x \in \Delta, \nu < r < \nu + 1], \\ \Delta_\nu^2 &= [x; x \in \Delta, (\nu+1)^{-1} < | x - F | < \nu^{-1}], \\ \Delta_0 &= [x; x \in \Delta, r < 2 \text{ diameter of } F]. \end{aligned}$$

Now ϕ_j^ν vanishes off of $\Delta_\nu^1 \cup \Delta_\nu^2$. On Δ_ν^1 we have

$$\begin{aligned} \left| \int_{\Delta_\nu^1} \langle \phi^\nu \phi_j^\nu A^j z, z\rangle dx \right| &\leqq 2 \int_{\Delta_\nu^1} [\,| b | \, | \zeta^0 |^2 + | a^{ij} \phi_j^\nu \bar{\zeta}^i \zeta^0 | \,] dx \\ &\leqq 2 \int_{\Delta_\nu^1} [(| b | / e) e | \zeta^0 |^2 + 2(\bar{\alpha}/e)^{1/2} (a^{ij} \zeta^i \bar{\zeta}^j)^{1/2} (e | \zeta^0 |^2)^{1/2}] dx \\ &\leqq 2 \sup_{\Delta_\nu^1} [\,| b | / e] \int_{\Delta_\nu^1} e | \zeta^0 |^2 dx + 2 \sup_{\Delta_\nu^1} [\bar{\alpha}/e]^{1/2} \int_{\Delta_\nu^1} [e | \zeta^0 |^2 + a^{ij} \zeta^i \bar{\zeta}^j] dx. \end{aligned}$$

Hence if we employ the growth estimates H_4 we obtain

$$\left| \int_{\Delta_\nu^1} \langle \phi^\nu \phi_j^\nu A^j z, z\rangle dx \right| \leqq K_1 \nu \int_{\Delta_\nu^1} \langle Ez, z\rangle dx. \tag{3.11}$$

As regards a Δ_ν^2 estimate, we note that it is clear from the form of M^* that ζ^0 is strongly differentiable on Δ_0. Thus the argument employed in [6, §5] shows that

$$\left| \int_{\Delta_\nu^2} \langle \phi^\nu \phi_j^\nu A^j z, z\rangle dx \right| \leqq K_2 \left[\nu \log \nu \int_{\Delta_0} | \nabla \zeta^0 |^2 dx \int_{\Delta_\nu^2} \langle Ez, z\rangle dx \right]^{1/2}. \tag{3.12}$$

Combining the inequalities (3.10), (3.11), and (3.12) we have for sufficiently large ν

$$\epsilon \leqq K_3 \left\{ \nu \int_{\Delta_\nu^1} \langle Ez, z\rangle dx + \left[\nu \log \nu \int_{\Delta_\nu^2} \langle Ez, z\rangle dx \right]^{1/2} \right\}.$$

Hence either

$$\int_{\Delta_\nu^1} \langle Ez, z\rangle dx \geqq \frac{\epsilon}{2K_3} \frac{1}{\nu}$$

or

$$\int_{\Delta_\nu'} \langle Ez, z\rangle dx \geqq \left[\frac{\epsilon}{2K_3}\right]^2 \frac{1}{\nu \log \nu}$$

and it follows that

$$\int_{\Delta} \langle Ez, z\rangle dx \geqq \min\left[\frac{\epsilon}{2K_3}, \left(\frac{\epsilon}{2K_3}\right)^2\right] \sum \frac{1}{\nu \log \nu} = \infty,$$

which is contrary to z belonging to $L_2(\Delta, E)$. This concludes the proof of Lemma 3.1.

An analogous argument applies to the adjoint operator

$$Nz = E^{-1}[-(A^i z)_i + (D - B)z]$$

with domain consisting of all complex vector-valued functions $z = (u, v^1, \cdots, v^m)$ which are piecewise smooth in $\bar{\Delta}$, have bounded supports, and satisfy the boundary condition

$$\sigma(a^{ni}v^i + bu) - \tau u = 0, \qquad x \in \dot{\Delta}. \tag{3.13}$$

It follows that $\overline{N}$ is maximal dissipative. On the other hand the general theory of dissipative operators (see [10]) implies that M^* is also maximal dissipative. Since N is clearly a restriction of M^*, it follows that $\overline{N} = M^*$. This proves

COROLLARY. *Under the hypotheses* H_1, H_2, H_3, H_4, *and* H_5, *the operator* $\overline{N}$ *is maximal dissipative with dense domain,* $\overline{N} = M^*$, *and* $\overline{M} = N^*$.

Proof of Theorem 2.1. We now obtain the operator L as a retract of $\overline{M}$. To this end we define the projection operator

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

associated with the subspace h_1 of $L_2(\Delta, E)$ having all but the first coordinate function identically zero. Clearly h_1 is equivalent with $L_2(\Delta, e)$. We next define the restriction $\overline{M}'$ of $\overline{M}$ by

$$\mathfrak{D}(\overline{M}') = [y;\, y \in \mathfrak{D}(\overline{M}),\, \overline{M}y \in h_1],$$

and the retraction of $\overline{M}$ to h_1, which we denote by $\overline{M}''$, as

$$\mathfrak{D}(\overline{M}'') = [P_1 y;\, y \in \mathfrak{D}(\overline{M}')],\quad \overline{M}'' P_1 y = \overline{M}' y.$$

We note that $y \in \mathfrak{D}(\overline{M})$ belongs to $\mathfrak{D}(\overline{M}')$ if and only if u is strongly differentiable and $v^i = u_i$ for each i. Because of (3.5) we have

$$\begin{aligned} (My, y) + (y, My) &= \int_{\dot{\Delta}} \langle A^n y, y\rangle dS + \int_{\Delta} \langle Dy, y\rangle dx \\ &\leqq \int_{\Delta} \langle Dy, y\rangle dx \leqq -(P_2 y, y). \end{aligned} \tag{3.14}$$

This holds as well for $\overline{M}$ so that condition (4.1) of [11] is valid. This being so, it is proved in [11] that $\overline{M}''$ is maximal dissipative with dense domain in h_1. Moreover it is obvious that L as defined in Definition 2.2 is precisely $\overline{M}''$. On the other hand $\overline{M}=N^*$ implies that $L=(N^*)''$. However this is just the definition of L as given in Definition 2.1. This completes the proof of Theorem 2.1 and shows, incidentally, that L is maximal dissipative with dense domain in $L_2(\Delta, e)$.

REMARK. In view of (3.14) we see that $\overline{M}+P_2/2$ is dissipative and hence that

$$\frac{1}{2}P_1 - \overline{M} = \frac{1}{2}I - (\overline{M} + \frac{1}{2}P_2)$$

has an inverse of bound 2 (see [10]). Actually $\overline{M}+P_2/2$ is maximal dissipative since if it had a proper dissipative extension, say R, then $R-P_2/2$ would be a proper dissipative extension for $\overline{M}$, contrary to $\overline{M}$ being maximal dissipative. It follows that the range of $(P_1/2-\overline{M})$ is all of $L_2(\Delta, E)$.

4. **L_2-positivity.** As stated in the introduction, a necessary and sufficient condition for the semi-group of operators generated by L to be positive is that the resolvent $R(\lambda, L)$ be positive for λ sufficiently large. The positivity part of Theorem 2.2 therefore requires that we prove

LEMMA 4.1. *Suppose the conditions* H_1, H_2, H_3, H_4, *and* H_5 *are satisfied and let L be the operator defined as in Definition* 2.2. *Then* $R(\lambda, L)$ *is positive for all* $\lambda>0$.

Proof. Let $f\geqq 0$ in $L_2(\Delta, e)$ be given. Then since L is maximal dissipative with dense domain, for each $\lambda>0$ there is a unique $u\in\mathfrak{D}(L)$ with

$$\lambda u - Lu = f. \tag{4.1}$$

The function u will be real-valued since otherwise its real part would serve as well, contrary to the uniqueness of u. We must show that u is almost everywhere non-negative. Let

$$\Delta_0 = [x;\, x \in \Delta,\, u(x) < 0]. \tag{4.2}$$

Setting $y=(u, v^1, \cdots, v^m)$ where $v^i=u_i$, it is clear from the proof of Theorem 2.1 that $y\in\mathfrak{D}(\overline{M}')\subset\mathfrak{D}(\overline{M})$. We shall first show that

$$(\overline{M}y, y)_{\Delta_0} + (y, \overline{M}y)_{\Delta_0} \leqq 0. \tag{4.3}$$

To this end we define the function $\rho(s)$ of class C^2 on the reals to the reals such that $\rho(s)=1$ for $s\leqq -1$, $\rho(s)=0$ for $s\geqq 0$, $0\leqq\rho'(s)\leqq 2$ and set

$$\omega^\nu(x) = \rho[\nu u(x)], \qquad x \in \Delta.$$

We see that

$$\omega_i^\nu = \nu\rho'[\nu u]u_i$$

and since this vanishes for $u(x) \leqq -1/\nu$ and $u(x) \geqq 0$, we have

$$|u|\,|\overset{\nu}{\omega}_i| \leqq 2|u_i|. \tag{4.4}$$

The functions ω^ν have strong first partials with u. Let $y^k = (u^k, v^{ik})$ approximate y in the sense of Definition 2.2. Then

$$((\omega^\nu)^2 My^k, y^k) + ((\omega^\nu)^2 y^k, My^k) = [(M\omega^\nu y^k, \omega^\nu y^k) + (\omega^\nu y^k, M\omega^\nu y^k)]$$
$$- [(A^i \overset{\nu}{\omega}_i \overset{k}{y}, \overset{\nu}{\omega} \overset{k}{y}) + (\overset{\nu}{\omega} \overset{k}{y}, A^i \overset{\nu}{\omega}_i \overset{k}{y})].$$

It is clear that $\omega^\nu y^k$ satisfies the boundary condition (i) of Definition 2.2 along with y^k. Thus if ω^ν were continuously differentiable an integration by parts would shows that the first bracket expression on the right is nonpositive. However, since ω^ν is strongly differentiable, it can be approximated by smooth functions in such a way that the corresponding bracket expression will also converge to a nonpositive limit. Next passing to the limit as k tends to ∞, we get

$$((\overset{\nu}{\omega})^2 \overline{M}y, y) + ((\overset{\nu}{\omega})^2 y, \overline{M}y) \leqq - [(A^i \overset{\nu}{\omega}_i y, \overset{\nu}{\omega} y) + (\overset{\nu}{\omega} y, A^i \overset{\nu}{\omega}_i y)].$$

We proceed to estimate the terms in the right member.

$$|(A^i \overset{\nu}{\omega}_i y, \overset{\nu}{\omega} y)| \leqq 2\int_\Delta [|\overset{\nu}{\omega} b^i \overset{\nu}{\omega}_i u^2| + |\overset{\nu}{\omega} a^{ij} \overset{\nu}{\omega}_i u_j u|]dx$$
$$\leqq 4\int_{-1/\nu<u<0} [|b^i u_i u| + |a^{ij} u_i u_j|]dx$$

by (4.4). Making use of condition H_4 we see that

$$|b^i u_i u| \leqq \frac{(a^{ij}u_i u_j)^{1/2}}{\alpha} \frac{|b|}{e^{1/2}} |e^{1/2}u| \leqq K[e|u|^2 + a^{ij}u_i u_j]$$

and therefore

$$|(A^i \overset{\nu}{\omega}_i y, \overset{\nu}{\omega} y)| \leqq 4(K+1)\int_{-1/\nu<u<0} [e|u|^2 + a^{ij}u_i u_j]dx.$$

Since $(y, y) < \infty$, it is clear that this expression tends to zero with $1/\nu$. On the other hand $-\omega^\nu$ tends boundedly to the characteristic function of Δ_0. Hence

$$(\overline{M}y, y)_{\Delta_0} + (y, \overline{M}y)_{\Delta_0} \leqq \lim_\nu - [(A^i \overset{\nu}{\omega}_i y, \overset{\nu}{\omega} y) + (\overset{\nu}{\omega} y, A^i \overset{\nu}{\omega}_i y)] = 0.$$

We recall that Lu is the first component of $\overline{M}y$, the other components vanishing identically. As a consequence (4.1) implies

$$2\lambda|u|^2_{\Delta_0} \leqq 2\lambda|u|^2_{\Delta_0} - [(\overline{M}y, y)_{\Delta_0} + (y, \overline{M}y)_{\Delta_0}] = (f, u)_{\Delta_0} + (u, f)_{\Delta_0} \leqq 0$$

since $u<0$ and $f\geqq 0$ on Δ_0. This proves that Δ_0 is of measure zero and hence that $u\geqq 0$ almost everywhere in Δ.

5. L_2-**analyticity.** All parts of Theorem 2.2 have been established except for the analyticity of the semi-group of operators generated by L and we now proceed to make up for this deficiency. The general pattern of our argument follows a proof due to K. Yosida [15] for a very much restricted initial-value problem of this type. Basic to the proof is the following lemma.

LEMMA 5.1. *Suppose the conditions* H_1, H_2, H_3, H_4, *and* H_5 *are satisfied and let L be the operator defined as in Definition* 2.2. *Then*

$$| [(\mu + i\nu)I - L]u | \geqq 2^{-1/2}(\mu + |\nu|)|u|, \tag{5.1}$$

for all $u\in\mathfrak{D}(L)$ *and real* μ, ν *with* $\mu>0$.

Proof. Suppose that $u\in\mathfrak{D}(L)$ and set $y=(u, v^i)$ where $v^i=u_i$. Then as we have seen in §3, $y\in\mathfrak{D}(\overline{M})$. If y were smooth with a bounded support then

$$\begin{aligned}(P_1My, y) &= \int_{\dot{\Delta}}(a^{ni}v^i + b^n u)\bar{u}dS \\ &\quad - \int_{\Delta}[a^{ij}v^j u_{\bar{i}} + b^i u u_{\bar{i}} - b^i(u_i - v^i)\bar{u} - c|u|^2]dx.\end{aligned} \tag{5.2}$$

Now the vector-valued functions in the approximating sequence $\{y^k\}$ of Definition 2.2 are smooth with bounded support and since each y^k satisfies the boundary conditions we have

$$\int_{\dot{\Delta}}(a^{ni}v^{ik} + b^n u^k)\overline{u^k}dS \leqq 0.$$

Further, $My^k\to(Lu, 0, \cdots, 0)$, $(u^k, u^k_i)\to(u, u_i)$, and $(u^k, v^{ik})\to(u, u_i)$ all in the $L_2(\Delta, E)$ topology. Since by H_3 $|c|\leqq Ke$ and by H_4

$$|b^i u_i u| \leqq \frac{|b|}{\alpha e^{1/2}}(a^{ij}u_i u_{\bar{j}})^{1/2}(e|u|^2)^{1/2} \leqq K[a^{ij}u_i u_{\bar{j}} + e|u|^2],$$

we obtain, on replacing y by y^k in (5.2) and passing to the limit as k tends to ∞,

$$(Lu, u) + \int_{\Delta}(a^{ij}u_i u_{\bar{j}} + b^i u u_{\bar{i}} - c|u|^2)dx \leqq 0. \tag{5.3}$$

Applying condition H_3 we see that

$$\text{re}[2b^i u u_{\bar{i}}] \leqq a^{ij}u_i u_{\bar{j}} - c|u|^2,$$

so that for $\mu>0$,

$$\mathrm{re}([(\mu + i\nu)I - L]u, u) \geqq \mu | u |^2 + \frac{1}{2}\int_\Delta [a^{ij}u_i u_{\bar{j}} - c | u |^2] dx.$$

On the other hand,

$$| \mathrm{im}(Lu, u) | = \left| \mathrm{im}\left\{ - \int_\Delta b^i u u_{\bar{i}} dx \right\} \right| \leqq \frac{1}{2}\int_\Delta [a^{ij}u_i u_{\bar{j}} - c | u |^2] dx$$

and hence

$$| \mathrm{im}([(\mu + i\nu)I - L]u, u) | \geqq | \nu | \, | u |^2 - \frac{1}{2}\int_\Delta [a^{ij}u_i u_{\bar{j}} - c | u |^2] dx.$$

Combining these estimates we have

$$\begin{aligned} 2^{1/2} | ([(\mu + i\nu)I - L]u, u) | & \\ &\geqq | \mathrm{re}([(\mu + i\nu)I - L]u, u | + | \mathrm{im}([(\mu + i\nu)I - L]u, u | \\ &\geqq (\mu + | \nu |) | u |^2 \end{aligned}$$

which clearly implies (5.1).

The analyticity of $S(t)$ for $t>0$ now follows from a result due to K. Yosida [14]. However, we can do somewhat better and show that $S(t)$ is actually analytic in the sector $-\pi/4<\arg t<\pi/4$. Since L generates a strongly continuous semi-group of contraction operators, it is known (see [4, Theorem 11.5.2]) that the resolvent set of L contains the right half plane. Hence Lemma 5.1 implies

$$| R(\mu + i\nu; L) \leqq 2^{1/2}(\mu + | \nu |)^{-1}, \qquad \text{for } \mu > 0.$$

We can now continue $R(\lambda; L)$ into the left half plane by analytic continuation. In fact, for $|\lambda - i\nu| < 2^{-1/2}|\nu|$ we have (see [4, Theorem 5.83 and 5.91])

$$R(\lambda; L) = \sum_{n=0}^{\infty} (i\nu - \lambda)^n [R(i\nu; L)]^{n+1}$$

form which it follows that

$$| R(\lambda; L) | \leqq 2^{1/2}(| \nu | - 2^{1/2} | \lambda - i\nu |)^{-1}.$$

This inequality can be given a simple geometric meaning. Let Σ denote the sector $3\pi/4 \leqq \arg \lambda \leqq 5\pi/4$. Then the above estimate gives for each λ outside of Σ

$$| R(\lambda; L) | \leqq 2^{1/2}/d(\lambda)$$

where $d(\lambda)$ is the distance from λ to Σ. It now follows directly from [4, Theorem 12.8.1] that $S(t)$ is analytic in the sector $-\pi/4<\arg t<\pi/4$.

Combining this result with the material in §§3 and 4 we obtain a proof of Theorem 2.2 in its entirety.

6. **L_1-theory.** The previous material can be adapted so as to provide us with a positivity preserving solution to (1.1) with the boundary conditions (1.3) in the space $L_1(\Delta, e)$. We show first of all that a sufficiently large subset of $\mathfrak{D}(L)$ lies in $L_1(\Delta, e)$. If Δ is bounded this is no problem since all of $\mathfrak{D}(L)$ is then contained in $L_1(\Delta, e)$. However, for unbounded Δ we require a slightly more stringent growth condition on the coefficients than H_4 in order to obtain the desired result.

LEMMA 6.1. *Suppose the conditions H_1, H_2, H_3, H_4', and H_5 are satisfied and let L be the operator defined as in Definition 2.2. For fixed $\lambda \geqq 1/2$ and $u \in \mathfrak{D}(L)$ suppose that $f = \lambda u - Lu$ has a bounded support. Then u and Lu lie in $L_1(\Delta, e)$. Further setting $\phi^k = 1$ for $r \leqq k$, $\phi^k = 1-(r-k)$ for $k<r<k+1$, and $\phi^k = 0$ for $r \geqq k+1$, then $\phi^k u \in \mathfrak{D}(L)$ and*

$$[\phi^k u, L\phi^k u] \to [u, Lu] \quad \text{in} \quad L_1(\Delta, e) \times L_1(\Delta, e). \tag{6.1}$$

Proof. Let $y \in (u, v^i)$ with $v^i = u_i$ and let $F = (f, 0, \cdots, 0)$. Then $y \in \mathfrak{D}(\overline{M}')$ and if θ is any ultimately constant smooth scalar-valued function, then it is readily verified using the approximating sequence $\{y^k\}$ of Definition 2.2 that $\theta y \in \mathfrak{D}(\overline{M})$ and in fact that

$$\overline{M}\theta y = \theta \overline{M} y + \theta_i A^i y. \tag{6.2}$$

Suppose that the support of f is contained in the sphere $r < r_0$. We define a family of such θ functions as follows:

$$\begin{aligned} \theta^\nu(r) &= 1, & r \leqq r_0, \\ d\theta^\nu/dr &= \begin{cases} 2\gamma(r^{-1}\log r)\theta, \\ 0, \end{cases} & \begin{matrix} r_0 \leqq r \leqq \nu, \\ \nu \leqq r, \end{matrix} \end{aligned} \tag{6.3}$$

where γ is a positive constant to be determined presently. For $r > r_0$ we see that

$$\theta(r) \equiv \lim_{\nu \to \infty} \theta^\nu(r) = K \exp[\gamma(\log r)^2].$$

In order to avoid working with the unbounded θ, we make use of the θ^ν's and at an appropriate time pass to the limit.

Now

$$| (\theta_i^\nu A^i y, \theta^\nu y) | \\ \leqq 2 \int_{r_0<r<\nu} \left[\frac{(a^{ij}\theta_i^\nu \theta_j^\nu)^{1/2}}{e^{1/2}\theta^\nu} \cdot (a^{ij}\theta^\nu v^i(\theta^\nu v^j)^-)^{1/2} e^{1/2} | \theta^\nu u | + \frac{| b^i \theta_i^\nu |}{e\theta^\nu} e |\theta^\nu u|^2 \right] dx.$$

Applying condition H_4' together with (6.3) we see that γ can be chosen sufficiently small so that

$$(6.4)\qquad |(\theta_i^\nu A^i y, \theta y)| \leqq \frac{1}{4} |\theta^\nu y|_2^2, \qquad \nu > r_0.$$

On the other hand (6.2) implies

$$\left(\lambda - \frac{1}{2}\right) P_1 \theta^\nu y + \left[\frac{1}{2} P_1 \theta^\nu y - \overline{M} \theta^\nu y\right] = \theta^\nu F - \theta_i^\nu A^i y$$

and making use of (3.14) and the subsequent remark, we see that

$$(\theta^\nu y, \theta^\nu y) \leqq (\theta^\nu F, \theta^\nu y) + (\theta^\nu y, \theta^\nu F) - [(\theta_i^\nu A^i y, \theta^\nu y) + (\theta^\nu y, \theta_i^\nu A^i y)].$$

Since $\theta^\nu F = F$, the estimate (6.4) gives $|\theta^\nu y|_2 \leqq 4|F|_2$. Passing to the limit as ν tends to infinity we obtain the basic inequality

$$(6.5)\qquad |\theta y|_2 \leqq 4|F|_2.$$

As a first consequence of (6.5) we have

$$(6.6)\qquad \begin{aligned}\int_\Delta [e|u| + (a^{ij}v^i v^{j-})^{1/2}]dx &\leqq \left\{\int_\Delta [e|\theta u|^2 + a^{ij}\theta v^i(\theta v^j)^-]dx\right\}^{1/2} \\ &\quad \cdot \left\{\int_\Delta [e\theta^{-2} + \theta^{-2}]dx\right\}^{1/2},\end{aligned}$$

which is finite by virtue of (6.5) and the fact that θ increases faster than any power of r. Thus in particular $u \in L_1(\Delta, e)$ and it is obvious that $\phi^k u \to u$ in $L_1(\Delta, e)$. Moreover

$$L\phi^k u = \lambda \phi^k u - \phi^k f + e^{-1}[2a^{ij}\phi_i^k u_j - b^i \phi_i^k u + a^{ij}\phi_{ij}^k u + a_j^{ij}\phi_i^k u].$$

It is clear that the first two terms on the right converge to λu and f, respectively, in the $L_1(\Delta, e)$ metric. Hence (6.1) will be proved if we show that the bracket term tends to zero in $L_1(\Delta, e)$. Now

$$(6.7)\qquad \begin{aligned}\int_\Delta |a^{ij}\phi_i^k u_j| dx &\leqq \int_\Delta (a^{ij}\phi_i^k \phi_j^k)^{1/2}(a^{ij}u_i u_{\bar{j}})^{1/2} dx \\ &\leqq \left[\int_{k<r<k+1} \frac{\bar{\alpha}}{e}\frac{e}{\theta^2} dx \int_{k<r<k+1} \theta^2 a^{ij}u_i u_{\bar{j}} dx\right]^{1/2}.\end{aligned}$$

Both of the integrands in the right member are integrable over Δ, the first because of H_4' and the definition of θ, the second because of (6.5). It follows that the left member tends to zero with $1/k$. Likewise

$$\int_\Delta |b^i \phi_i^k u| dx \leqq \left[\int_{k<r<k+1} \left(\frac{|b|}{e}\right)^2 \frac{e}{\theta^2} dx \int_{k<r<k+1} e|\theta u|^2 dx\right]^{1/2}$$

tends to zero with $1/k$. Again $\phi_{ij}^k = O(r^{-1})$ so that

$$\int_\Delta |a^{ij}\phi_{ij}^k u|\,dx \leqq \left[\int_{k<r<k+1} \frac{\bar{\alpha}}{e}\theta^{-2}dx \int_{k<r<k+1} e|\theta u|^2 dx\right]^{1/2}.$$

Finally

$$\int_\Delta |a_j^{ij}\phi_i^k u|\,dx \leqq \left[\int_{k<r<k+1} \frac{|a_j^{ij}|}{e}\theta^{-2}dx \int_{k<r<k+1} e|\theta u|^2 dx\right]^{1/2}.$$

Applying H_4' we see that the last two terms in the bracket also tend to zero with $1/k$ in $L_1(\Delta, e)$. This completes the proof of Lemma 6.1.

The previous lemma will be used to show that the range of $(\lambda I - L)$ fills out $L_1(\Delta, e)$. Another essential ingredient is a bound for $(\lambda I - L)^{-1}$. To obtain such a bound it suffices to consider only non-negative functions in $\mathfrak{D}(L)$. The following lemmas supply the necessary estimates for the bound of $(\lambda I - L)^{-1}$.

LEMMA 6.2. *Suppose that conditions* H_1, H_2, H_3, H_4' *and* H_5 *are satisfied and let L be the operator defined as in Definition* 2.2. *If* $\lambda \geqq 1/2$, *if* $u \in \mathfrak{D}(L)$ *is non-negative, and if* $f = \lambda u - Lu$ *has bounded support, then*

$$\int_{u>0} eLu\,dx \leqq \bar{c}\int_\Delta eu\,dx, \tag{6.8}$$

where $\bar{c} = \sup[c(x)/e(x);\, x \in \Delta]$.

Proof. Setting $y = (u, v^i)$ where $v^i = u_i$, we see by Definition 2.2 that there exists an approximating sequence $\{y^k\}$ of smooth real vector-valued functions with bounded support satisfying the boundary conditions and such that

$$[y^k, My^k] \to [y, \overline{M}y] \qquad \text{in } L_2(\Delta, E) \times L_2(\Delta, E).$$

Given the sequence $\{\phi^\nu\}$ of smooth scalar functions defined as in Lemma 6.1, it is clear that

$$\lim_{k\to\infty} [\phi^\nu y^k, M\phi^\nu y^k] = [\phi^\nu y, \overline{M}\phi^\nu y] \qquad \text{in } L_2(\Delta, E) \times L_2(\Delta, E).$$

Since each ϕ^ν has a bounded support, it follows, in particular, that the first component of $M\phi^\nu y^k$ converges in $L_1(\Delta, e)$. Now the first component of $\overline{M}\phi^\nu y$ is

$$\begin{aligned} P_1\overline{M}\phi^\nu y &= e^{-1}[(a^{ij}\phi^\nu v^j)_i + 2(b^i\phi^\nu u)_i - b^i\phi^\nu v^i + (c - b_i^i)\phi^\nu u] \\ &= L\phi^\nu u - (a^{ij}\phi_j^\nu u)_i + b^i\phi_i^\nu u. \end{aligned}$$

We have already shown in the proof of Lemma 6.1 that the last two terms on the right tend to zero in $L_1(\Delta, e)$ as $\nu \to \infty$. Consequently $\lim_\nu P_1\overline{M}\phi^\nu y = Lu$ in $L_1(\Delta, e)$ and it therefore suffices to show that

$$\lim_{\nu\to\infty}\lim_{k\to\infty}\int_{u>0} eP_1M\phi^\nu y^k \leqq \bar{c}\int_\Delta eudx. \tag{6.9}$$

We now define the function $\rho(s)$ of class C^2 on the reals to the reals such that $\rho(s)=0$ for $s\leqq 0$, $\rho(s)=1$ for $s\geqq 1$, and $0\leqq\rho'(s)\leqq 2$ elsewhere. Setting

$$\omega^{\mu k}(x) = \rho(\mu u^k(x)),$$

we see as in the proof of Lemma 4.1 that $\omega_i^{\mu k}=\mu\rho'(\mu u^k)u_i^k$ and

$$|u^k|\,|\omega_i^{\mu k}| \leqq 2|u_i^k|. \tag{6.10}$$

An integration by parts gives

$$\int_\Delta e(P_1M\phi^\nu y^k)\omega^{\mu k}dx = \int_{\dot\Delta}\phi^\nu(a^{nj}v^{jk}+b^n u^k)\omega^{\mu k}dS$$
$$-\int_\Delta\{a^{ij}\phi^\nu v^{jk}\omega_i^{\mu k}+b^i\phi^\nu u^k\omega_i^{\mu k}-(b^i\phi^\nu u^k)_i\omega^{\mu k}$$
$$+b^i\phi^\nu v^{ik}\omega^{\mu k}+b_i\phi^\nu u^k\omega^{\mu k}-c\phi^\nu u^k\omega^{\mu k}\}dx.$$

We show first of all that the surface integral is nonpositive. In fact, $\omega^{\mu k}(x)=0$ whenever $u^k(x)\leqq 0$ and hence in particular at points of $\dot\Delta$ at which $\sigma=0$. At those points of $\dot\Delta$ at which $\sigma\neq 0$ and $u^k>0$, we see from the boundary conditions that $(a^{nj}v^{jk}+b^n u^k)<0$. Since ϕ^ν and $\omega^{\mu k}$ are each non-negative, this proves that the surface integral is nonpositive.

We may now pass to the limit as $k\to\infty$. The volume integrals cause no difficulty and making use of the fact that $v^{ik}\to u_i$, we get

$$\int_\Delta e(P_1\overline{M}\phi^\nu y)\omega^\mu dx \leqq -\int_\Delta \mu\rho'(\mu u)\phi^\nu a^{ij}u_ju_idx+\int_\Delta c\phi^\nu u\omega^\mu dx$$
$$+\int_\Delta b^i\phi_i^\nu u\omega^\mu dx-\int_\Delta b^i\phi^\nu u\omega_i^\mu dx,$$

where we have set $\omega^\mu(x)=\rho(\mu u(x))$. Now the first integral on the right is clearly nonpositive. The second integral is less than or equal to $\bar{c}\int\phi^\nu eu\omega^\mu dx$. As in the proof of Lemma 6.1, the third integral tends to zero with $1/\nu$, uniformly in μ. Finally because of (6.10) the last integral is bounded by

$$2\int_{0<u<1/\mu}|b|\,\phi^\nu|\nabla u|\,dx$$

and hence goes to zero with $1/\mu$. Further we note that ω^μ converges boundedly to the characteristic function of the set $[x;\, x\in\Delta,\, u(x)>0]$. Thus passing to the limit, first with respect to μ and then ν we obtain

$$\lim_{\nu\to\infty}\int_{u>0} eP_1\overline{M}\phi^\nu y dx = \lim_{\nu\to\infty}\lim_{\mu\to\infty}\int_\Delta e(P_1\overline{M}\phi^\nu y)\omega^\mu dx \leqq \bar{c}\int_\Delta eudx,$$

which was to be proved.

Lemma 6.2 is not quite strong enough since it omits consideration of the set on which u vanishes. We have been able to get around this by requiring an additional smoothness condition on the coefficients.

LEMMA 6.3. *Suppose condition* H_2' *is satisfied and let* L *be the operator defined as in Definition* 2.2. *If* $u\in\mathfrak{D}(L)$ *and* $f=\lambda u-Lu$ *is Hölder continuous in the interior of* Δ, *then* u *is of class* C^2 *and its second partials are Hölder continuous in the interior of* Δ.

Proof. It is clear from Definition 2.2 that u satisfies the relation $f=\lambda u-Lu$ in the weak sense and has strong first partial derivatives in Δ. The ellipticity of L can now be employed (see L. Nirenberg [9, Lemma 1, §4]) to show that u has strong second partials, at least locally in Δ.(8) According to a result of C. B. Morrey [8, Theorem 4.6], this suffices to prove that u satifies the assertion of the lemma.

The above result has also been established by F. Browder [1], assuming c to have Hölder continuous first partials. The result also follows from the cruder Hilbert space methods (see [9]) if it is assumed that the coefficients are sufficiently smooth.

Added in proof.

LEMMA 6.4. *Suppose* $u(x)$ *is non-negative and has Hölder continuous second partial derivatives in a domain* Δ. *Then the second partials of* $u(x)$ *vanish almost everywhere on*

$$C_0 \equiv [x;\, u(x) = 0] \cap \Delta.$$

Proof. It suffices to show that this is true for compact subsets of C_0. Let C be such a subset. Since $u(x)\geqq 0$ in Δ, each point of C is a minimum for u and hence its first partials vanish on C. Thus the Taylor's expansion for $u(x)$ about an arbitrary point x_0 of C is of the form

$$u(x) = u_{ij}(\bar{x})(x^i - x_0^i)(x^j - x_0^j),$$

where $\bar{x}$ is some point on the line segment joining x_0 and x. The second partials are Hölder continuous on C, say with Hölder coefficient β, and therefore

$$u(x) = u_{ij}(x_0)(x^i - x_0^i)(x^j - x_0^j) + O(|x - x_0|^{2+\beta}).$$

Since x_0 is a minimum of $u(x)$, the matrix $(u_{ij}(x_0))$ is positive. If this matrix is not identically zero, there is a nonempty cone inside which the form is

(8) In the above cited Nirenberg lemma, it is convenient to include the cu term with f so as not to require c to be of class C^1.

greater than $\gamma|x-x_0|^2$ for some positive γ.[4] Because of the order of the remainder term it is seen that in a sufficiently small sphere about x_0 the function $u(x)$ can have no zeros interior to the above cone. It follows that the density of zeros about x_0 will be less than one unless the matrix $(u_{ij}(x_0))$ vanishes. However any measurable set is of density one at all of its points with the possible exception of a set of measure zero. As a consequence the second partials of $u(x)$ must vanish almost everywhere on C.

COROLLARY. *Suppose conditions* H_1, H_2', H_3, H_4', *and* H_5 *are satisfied and let L be the operator defined in Definition* 2.2. *Suppose in addition that* $\lambda \geqq \frac{1}{2}$, $u \in \mathfrak{D}(L)$ *is non-negative, and that* $f=\lambda u - Lu$ *has bounded support and is Hölder continuous in the interior of* Δ. *Then*

$$\int_{C_0} eLudx = 0. \tag{6.11}$$

Proof. According to Lemma 6.3, the function $u(x)$ will have Hölder continuous second partial derivatives in Δ. On the set C_0 defined as above, $u(x)$ and its first partials automatically vanish and the previous lemma shows that the second partials vanish almost everywhere. The relation (6.11) now follows.

Proof of Theorem 2.3. To begin with let $\lambda \geqq 1/2$ and choose $f \geqq 0$ to be Hölder continuous on $\bar{\Delta}$ with bounded support. According to Theorem 2.2 there exists a non-negative $u \in \mathfrak{D}(L)$ such that $\lambda u - Lu = f$. Lemma 6.1 asserts that $u \in L_1(\Delta, e)$ so that $u \in \mathfrak{D}(K_0)$ and $\lambda u - K_0 u = f$. Applying Lemmas 6.2 and 6.4 we see that

$$\lambda \int_\Delta eudx = \int_\Delta e(Lu+f)dx \leqq \bar{c} \int_\Delta eudx + \int_\Delta efdx;$$

in other words

$$(\lambda - \bar{c})\,|u|_1 \leqq |f|_1. \tag{6.12}$$

Now any real-valued function f which is Hölder continuous on $\bar{\Delta}$ and has a bounded support will have positive and negative parts, f_+ and f_-, possessing these same properties; $|f|_1 = |f_+|_1 + |f_-|_1$. Applying the above result to f_+ and f_- in turn and adding gives (6.12) for f; and again the corresponding $u \in \mathfrak{D}(K_0)$.

On the other hand for any real-valued $u \in \mathfrak{D}(K_0)$ we may set $f = \lambda u - K_0 u$. Since $f \in L_1(\Delta, e) \cap L_2(\Delta, e)$ we can approximate f in both $L_1(\Delta, e)$ and $L_2(\Delta, e)$ by a sequence of smooth real-valued functions $\{f^n\}$ with bounded support. It follows from the above development that for each f^n there will exist a $u^n \in \mathfrak{D}(K_0)$ such that $\lambda u^n - K_0 u^n = f^n$, $(\lambda - \bar{c})|u^n|_1 \leqq |f^n|_1$, and $(\lambda - \bar{c})|u^n - u^k|_1$

[4] Diagonalizing $(u_{ij}(x_0))$ into the form $\sum_1^k \gamma^i (x^i)^2$, $\gamma^i > 0$, $1 < k \leqq m$, let $2\gamma = \min \gamma^i$. Then inside of the cone $\sum_1^k (\gamma^i - \gamma)(x^i)^2 \geqq \gamma \sum_{k+1}^m (x^i)^2$ we have $\sum_1^k \gamma^i (x^i)^2 \geqq \gamma r^2$.

$\leqq |f^n - f^k|_1$. However by Theorem 2.2, $\lambda|u^n - u|_2 \leqq |f^n - f|_2$ and we may therefore conclude that $u_n \to u$ in $L_1(\Delta, e)$ and hence that (6.12) holds for any real-valued function in $\mathfrak{D}(K_0)$.

In order to extend this result to all of $\mathfrak{D}(K_0)$ we proceed as follows. Lemma 6.1 shows that the range of $(\lambda I - K_0)$ contains in particular all real simple functions of bounded support. Suppose f is a complex-valued simple function, that is, suppose

$$f = \sum_{i=1}^{n} c^i \chi^i$$

where the c^i are complex-valued and the χ^i are characteristic functions of disjoint bounded measurable sets. As above, there will exist a real-valued $u^i \in \mathfrak{D}(K_0)$ such that $\lambda u^i - K_0 u^i = \chi^i$ and $(\lambda - \bar{c})|u^i|_1 \leqq |\chi^i|_1$. It is clear that $u = \sum c^i u^i \in \mathfrak{D}(K_0)$ and $\lambda u - K_0 u = f$. Further

$$(\lambda - \bar{c})\,|u|_1 \leqq (\lambda - \bar{c}) \sum |c^i|\ |u^i|_1 \leqq \sum |c^i|\ |\chi^i|_1 = |f|_1.$$

Another limiting process, similar to the one developed in the preceding paragraph, serves to extend (6.12) to all of $\mathfrak{D}(K_0)$.

Our next extension is to the smallest closed extension of K_0 which we denote by K. The usual argument using test functions shows that the closure of the differential operator K_0 is a well defined operator. It is clear that (6.12) continues to hold for all u in $\mathfrak{D}(K)$. Thus $(\lambda I - K)^{-1}$ exists for all $\lambda \geqq 1/2$ and is of norm $\leqq (\lambda - \bar{c})^{-1}$. As we have already noted the range of $(\lambda I - K_0)$ contains all simple functions and by Lemma 4.1, $(\lambda I - K_0)^{-1}$ is positive. Now the range of $(\lambda I - K)$ is closed with K and hence fills out $L_1(\Delta, e)$. Moreover since the simple functions are dense in the positive cone of $L_1(\Delta, e)$ it follows by continuity that $(\lambda I - K)^{-1}$ is positive. Thus K is a closed linear operator with dense domain whose resolvent exists for all $\lambda \geqq 1/2$, where it is positive and of norm $\leqq (\lambda - \bar{c})^{-1}$. The assertion of Theorem 2.3 now follows by virtue of the Hille-Yosida theorem [4, Theorems 11.7.2 and 12.3.1].

If all the boundary conditions are of the $\sigma > 0$ type, then the hypothesis H_2' in Theorem 2.3 can be replaced by H_2 since we can then prove the following lemma.

LEMMA 6.5. *Suppose conditions* H_1, H_2, H_3, H_4', *and* H_5 *are satisfied and in addition that* σ *is bounded away from zero on bounded subsets of* $\dot{\Delta}$. *Let* L *be the operator defined as in Definition* 2.2. *If* $\lambda \geqq 1/2$, *if* $u \in \mathfrak{D}(L)$ *is non-negative, and if* $f = \lambda u - Lu$ *has bounded support, then*

$$\int_\Delta eLudx \leqq \bar{c} \int_\Delta eudx,$$

where $\bar{c} = \sup [c(x)/e(x);\ x \in \Delta]$.

Proof. Proceeding as in the proof of Lemma 6.2 we see that it suffices to show that (see (6.9))

$$\lim_{\nu\to\infty}\lim_{k\to\infty}\int_{\Delta} eP_1M\phi^{\nu}y^k dx \leqq \bar{c}\int_{\Delta} eudx.$$

An integration by parts leads to

$$\int_{\Delta} eP_1M\phi^{\nu}y^k dx = \int_{\dot{\Delta}} (a^{ni}\phi^{\nu}v^{ik} + b^n\phi^{\nu}u^k)dS$$
$$+ \int_{\Delta} [c\phi^{\nu}u^k + b^i\phi^{\nu}(u_i^k - v^{ik}) + b^i\phi_{;i}^{\nu}u^k]dx.$$

Now the boundary condition implies that $a^{ni}v^{ik}+b^nu^k = -(\tau/\sigma)u^k$ on $\dot{\Delta}$. Further on the support of ϕ^{ν} there is no difficulty in passing to the limit in the second integral on the right. Hence

$$\lim_{k\to\infty}\int_{\Delta} eP_1M\phi^{\nu}y^k dx \leqq \limsup_{k\to\infty} -\int_{\dot{\Delta}}(\tau/\sigma)\phi^{\nu}u^k dS + \int_{\Delta} c\phi^{\nu}udx + \int_{\Delta} b^i\phi_{;i}^{\nu}udx.$$

As before the last term on the right tends to zero with $1/\nu$. It remains only to show that $\lim_k -\int_{\dot{\Delta}}(\tau/\sigma)\phi^{\nu}u^k dS \leqq 0$, in which case

$$\lim_{\nu\to\infty}\lim_{k\to\infty}\int_{\Delta} eP_1M\phi^{\nu}y^k \leqq \lim_{\nu\to\infty}\int_{\Delta} c\phi^{\nu}udx \leqq \bar{c}\int_{\Delta} eudx,$$

which is the desired result.

We recall that the strong first partials of u exist and that $y = (u, u_i)$ is approximated in $L_2(\Delta, E)$ by the vector-valued function (u^k, u_i^k). It follows from this that if we replace $u(x)$ by

$$\lim_{\epsilon\to\infty} (2\epsilon)^{-m}\int_{-\epsilon<x'^i<\epsilon} u(x - x')dx'$$

then the lim $u(x')$ as x' tends to $x\in\dot{\Delta}$ along a normal trajectory is equal to $\lim_k u^k(x)$, both of these limits being taken in the L_2-sense on $\dot{\Delta}$. These facts are readily established on writing $u(x)$ as the integral along a normal trajectory of its normal derivative. Since we have taken $u(x) \geqq 0$ in Δ, the same will be true of its assumed values on $\dot{\Delta}$. Finally we note that τ/σ is bounded on bounded subsets of $\dot{\Delta}$ and that ϕ^{ν} has a bounded support. We may therefore conclude that

$$\lim_{k\to\infty}\int_{\dot{\Delta}}(\tau/\sigma)\phi^{\nu}u^k dS = \int_{\dot{\Delta}}(\tau/\sigma)\phi^{\nu}udS \geqq 0.$$

This concludes the proof of Lemma 6.5.

Proof of the corollary to Theorm 2.3. We first show that the range of $(\lambda I - K_0)$ fills out $L_1(\Delta, e)\cap L_2(\Delta, e)$ for each $\lambda \geqq 1/2$. Hence suppose that $f\in L_1(\Delta, e)\cap L_2(\Delta, e)$. As in the proof of Theorem 2.3 we can approximate f by a sequence of smooth functions $\{f^n\}$ with bounded support which converge to f in both the $L_1(\Delta, e)$ and the $L_2(\Delta, e)$ metrics. For each f^n there will exist a $u^n \in \mathfrak{D}(K_0)$ such that $\lambda u^n - K_0 u^n = f^n$, $(\lambda - \bar{c})|u^n - u^m|_1 \leqq |f^n - f^m|_1$, $\lambda|u^n - u^m|_2 \leqq |f^n - f^m|_2$. Consequently $\{u^n\}$ converges in both $L_1(\Delta, e)$ and $L_2(\Delta, e)$ to a function u and $\{K_0 u^n\}$ likewise converges in both $L_1(\Delta, e)$ and $L_2(\Delta, e)$. Since L is closed $u \in \mathfrak{D}(L)$ and hence by Definition 2.3 $u \in \mathfrak{D}(K_0)$. Consequently $f = \lambda u - K_0 u$ which was to be shown. It now follows from a result in [7, Theorem 4.2] that the smallest closed extension K_p of K_0 considered as an operator in $L_p(\Delta, e)$, $1 < p < 2$, generates a strongly continuous semi-group of operators $[S(t)]$ with $|S(t)|_p \leqq \exp[(2/p - 1)\bar{c}t]$. We further note that for $\lambda \geqq 1/2$ the range of $(\lambda I - K_0)$ is dense in the positive cone of $L_p(\Delta, e)$. As a consequence $(\lambda I - K_p)^{-1}$ is positive with $(\lambda I - K_0)^{-1}$ and hence the semi-group operators $S(t)$ are also positive.

REMARK. The present development could just as well have been carried out on a Riemann manifold of class C^2. In fact Friedrichs in [3, §7] has prepared all of the necessary machinery for such a generalization.

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